

Entropy of Endomorphisms of Lie Groups

André Caldas* and Mauro Patrão†

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Abstract

We show, when G is a nilpotent or reductive Lie group, that the entropy of any surjective endomorphism coincides with the entropy of its restriction to the toral component of the center of G . In particular, if G is a semi-simple Lie group, the entropy of any surjective endomorphism always vanishes. Since every compact group is reductive, the formula for the entropy of an endomorphism of a compact group reduces to the formula for the entropy of an endomorphism of a torus. We also characterize the recurrent set of conjugations of linear semi-simple Lie groups.

1 Introduction

In [3] and [10], it is introduced a topological notion of entropy for proper maps on locally compact separable metrizable spaces. It is shown there that this topological entropy coincides with the supremum of the Kolmogorov-Sinai's entropies and also with the minimum of the Bowen's entropies. Using this variational principle, it is also shown in [10] that the topological entropy of a linear isomorphism of a finite dimensional vector space always vanishes. This shows that the Bowen's formula (see [2]) for the entropy of an endomorphism of a non-compact Lie Group gives just an upper bound for its topological entropy. At the end of [10], using again the variational principle, it is proved that the topological entropy of the endomorphism $\phi(z) = z^2$, where $z \in \mathbb{C}^*$, is equal to the topological entropy of its restriction to $S^1 \subset \mathbb{C}^*$. Using the

*Departamento de Matemática - Universidade de Brasília-DF, Brasil. Supported by CNPq grant no. 140888/11-0

†Departamento de Matemática - Universidade de Brasília-DF, Brasil. Supported by CNPq grant no. 310790/09-3

same kind of reasoning presented there, one can show that the same result holds for any endomorphism $\phi(z) = z^n$, where $n \in \mathbb{N}$.

These examples led us to the following conjecture. Since every connected abelian Lie group G is isomorphic to the product of a torus (the toral component of G) by a finite dimensional vector space and since the component of \mathbb{C}^* is just S^1 , the topological entropy of a proper endomorphism of a connected abelian Lie group G should be just the topological entropy of its restriction to the toral component of G .

More generally, for a connected Lie group G , we can consider the toral component of the identity component of the center of G , denoted by $T(G)$ and called in the present paper just *the toral component of G* . From now on we will use the term *entropy* to refer to the topological entropy. In this paper, when G is a nilpotent or reductive Lie group, we show that the entropy of any surjective endomorphism coincides with the entropy of its restriction to $T(G)$. Since every compact group is reductive, these results shed new light in the Bowen's formula even in the compact case. In fact, the formula for the entropy of an endomorphism of a compact group reduces to the formula for the entropy of an endomorphism of a torus (see [12]). In particular, if G is a compact semi-simple Lie group, the entropy of any surjective endomorphism always vanishes. One may wonder, for a general connected Lie group G , that the entropy of a surjective endomorphism coincides with the entropy of its restriction to $T(G)$. At the end of this article, we present arguments that suggest that the general case is slightly different.

The paper is structured in sections corresponding to the classes of Lie groups for which we compute the topological entropy of their respective endomorphisms. But first, in a preliminary section, we collect some results used in the remaining sections. Since the concept of topological entropy given in [10] requires properness, we always start the remaining sections considering surjective endomorphisms and showing that these endomorphisms are in fact proper maps. In Section 3, we treat the abelian case and shows the above conjecture about entropy. Section 4 treats the nilpotent case. As in the abelian case, we conclude that the entropy of an endomorphism of G is the entropy of its restriction to $T(G)$. In Section 5, the semi-simple case is considered. Using the relation in a linear semi-simple Lie group between endomorphisms, conjugations and linear maps, the recurrent set of a given conjugation is characterized. Then, the concept of Li-Yorke pairs is used to demonstrate that the entropy of endomorphisms of semi-simple Lie groups (even not linear ones) always vanishes. In particular, since $T(G)$ is trivial in this case, the entropy of an endomorphism coincides with the entropy of its restriction to $T(G)$. In Section 6, we compute the entropy of endomorphisms of reductive groups, by using the previous results for the

abelian and semi-simple cases. And finally, in Section 7, we end the paper with some remarks about the general case.

2 Preliminaries

In this section, we collect some facts that are used in the next sections. In general, we just state the facts with references, but we do not present much details of the theory involved. From [10], Theorem 3.2, we have the following variational principle, where $h(\phi)$ is the *topological entropy* defined in [10]. That is, $h(\phi)$ is the supremum of the entropies taken over all *admissible covers*.

Proposition 2.1. *Let X be a locally compact metrizable separable space and $\phi : X \rightarrow X$ a proper map. Then*

$$\sup_{\mu} h_{\mu}(\phi) = h(\phi) = \min_d h_d(\phi).$$

As a consequence of Proposition 2.1 we present the following formula for the entropy of products, which is well known in the compact case.

Proposition 2.2. *Let X and Y be locally compact metrizable separable spaces and $\phi : X \rightarrow X$, $\psi : Y \rightarrow Y$ proper maps. Then,*

$$h(\phi \times \psi) = h(\phi) + h(\psi).$$

Proof. From the variational principle, we have that

$$\begin{aligned} h(\phi) + h(\psi) &= \sup_{\mu} h_{\mu}(\phi) + \sup_{\nu} h_{\nu}(\psi) \\ &= \sup_{\mu, \nu} h_{\mu \times \nu}(\phi \times \psi) \leq h(\phi \times \psi). \end{aligned}$$

On the other hand, by the variational principle there exist metrics d_1 and d_2 such that

$$h(\phi) = h_{d_1}(\phi) \quad \text{and} \quad h(\psi) = h_{d_2}(\psi).$$

But Proposition 2.2.15 of [4] states that for the maximum distance d ,

$$h_d(\phi \times \psi) \leq h_{d_1}(\phi) + h_{d_2}(\psi).$$

Now, the variational principle leads us to

$$h(\phi \times \psi) \leq h_{d_1}(\phi) + h_{d_2}(\psi) = h(\phi) + h(\psi).$$

□

The following proposition, which is used in section 6, also generalizes a simple result from the compact case (see Proposition 2.1 of [10]).

Proposition 2.3. *Let X and Y be locally compact metrizable separable spaces, and consider the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\psi} & Y \end{array},$$

where ϕ , ψ and π are proper surjective maps. Then,

$$h(\psi) \leq h(\phi).$$

The following result is proved in Corollary 16 of [2].

Proposition 2.4 (Bowen's Formula). *If ϕ is an endomorphism of a Lie group G and d is a right invariant distance, then*

$$h_d(\phi) = \sum_{|\lambda|>1} \log |\lambda|,$$

where λ are the eigenvalues of ϕ' , the differential of ϕ at the identity, counted with multiplicity.

The following result about compact principal bundles is proved in Theorem 19 of [2].

Proposition 2.5. *Let $\pi : X \rightarrow Y$ be a compact metrizable G -principal bundle. Assume that $\phi : X \rightarrow X$, $\psi : Y \rightarrow Y$, and $\tau : G \rightarrow G$ are continuous maps so that $\psi \circ \pi = \pi \circ \phi$ and $\phi(xg) = \phi(x)\tau(g)$. Then*

$$h(\phi) = h(\psi) + h(\tau).$$

A tool we used in order to show that certain systems have zero entropy is the concept of Li-Yorke pairs. In this paper, we introduce a topological definition of a Li-Yorke pair, which is more adjusted to the non-compact case than the usual definition which uses \limsup and \liminf of the distances between $\phi^n(x)$ and $\phi^n(y)$ (see page 2 of [1]).

Definition 2.6 (Li-Yorke pair). *Given a continuous function $\phi : X \rightarrow X$, $(a, b) \in X \times X$ is called a Li-Yorke pair when there exist $c \in X$, $n_k \rightarrow \infty$ and $m_k \rightarrow \infty$ such that $(\phi^{n_k}(a), \phi^{n_k}(b)) \rightarrow (a, b)$ and $(\phi^{m_k}(a), \phi^{m_k}(b)) \rightarrow (c, c)$.*

We shall prove that positive entropy implies the existence of Li-Yorke pairs. A stronger version could be demonstrated along the lines of Theorem 2.3 in [1], but since we do not need that level of generality, we shall prove this simpler version.

Definition 2.7 (Pinsker σ -algebra and product measure). *For a ϕ -invariant measure μ , the Pinsker σ -algebra \mathcal{P} , is greatest sub- σ -algebra such that $h_{\mu|\mathcal{P}}(\phi) = 0$. The product of μ with itself over \mathcal{P} , denoted by λ_μ , is the measure on $X \times X$ such that, for any measurable sets $A, B \subset X$,*

$$\lambda_\mu(A \times B) = \int E(1_A|\mathcal{P})E(1_B|\mathcal{P}) d\mu.$$

Notice that if μ is ϕ -ergodic, then λ_μ is $\phi \times \phi$ -ergodic (see [6]). Also, notice that if $\mu(A) = 0$, then $A \in \mathcal{P}$.

Lemma 2.8. *Denote by Δ , the set of diagonal points in $X \times X$. That is, $\Delta = \{(x, x) \mid x \in X\}$. Then, for a ϕ -ergodic measure μ such that $h_\mu(\phi) > 0$,*

$$\lambda_\mu(\Delta) = 0.$$

Proof. Since λ_μ is ergodic and Δ is $\phi \times \phi$ -invariant, it is enough to prove that $\lambda_\mu(\Delta) \neq 1$. Suppose on the contrary, that $\lambda_\mu(\Delta) = 1$. Then, for any measurable $B \subset X$,

$$\int E(1_B|\mathcal{P})E(1_{B^c}|\mathcal{P}) d\mu = \lambda_\mu(B \times B^c) = 0.$$

And since the conditional expectation of a non-negative function is non-negative, we have that

$$E(1_B|\mathcal{P})E(1_{B^c}|\mathcal{P}) = 0 \tag{1}$$

almost everywhere. Let

$$A = \{x \in X \mid E(1_B|\mathcal{P}) \neq 0\}.$$

Then, $A \in \mathcal{P}$. Now,

$$\begin{aligned} \mu(B \setminus A) &= \int_{A^c} 1_B d\mu \\ &= \int_{A^c} E(1_B|\mathcal{P}) d\mu \\ &= 0. \end{aligned}$$

Likewise,

$$\begin{aligned}\mu(A \setminus B) &= \int_A 1_{B^c} d\mu \\ &= \int_A E(1_{B^c} | \mathcal{P}) d\mu \\ &= 0,\end{aligned}$$

where the last equality is due to the fact that when restricted to A , $E(1_B | \mathcal{P}) \neq 0$, and therefore, according to equation (1), $E(1_{B^c} | \mathcal{P}) = 0$ almost everywhere.

Using those relations, since \mathcal{P} is a sub- σ -algebra containing A , and containing also all measurable sets with null μ measure, we conclude that $B \in \mathcal{P}$. But since B was arbitrary, this would imply that $h_\mu(\phi) = 0$. The contradiction implies that $\lambda_\mu(\Delta) \neq 1$. \square

Lemma 2.9. *Let $Z = \text{supp}(\mu)$ and $W = \text{supp}(\lambda_\mu)$. Denote by Δ_Z the set of diagonal points (z, z) , with $z \in Z$. That is, $\Delta_Z = \{(z, z) \mid z \in Z\}$. Then,*

$$\Delta_Z \subset W.$$

Proof. If $(z, z) \in Z \times Z$ is not in the support of λ_μ , then, there exists an open neighborhood U of (z, z) , such that $\lambda_\mu(U) = 0$. In this case, there exists an open neighborhood A of z , such that $z \in A \times A \subset U$. Then,

$$\begin{aligned}\lambda_\mu(U) = 0 &\Rightarrow \lambda_\mu(A \times A) = 0 \\ &\Leftrightarrow \int E(1_A | \mathcal{P})^2 d\mu = 0 \\ &\Leftrightarrow \int E(1_A | \mathcal{P}) d\mu = 0 \\ &\Leftrightarrow \mu(A) = 0.\end{aligned}$$

But this contradicts the fact that z is in the support of μ . \square

Proposition 2.10. *Let X be a locally compact metrizable separable space and $\phi : X \rightarrow X$ a proper map. If $h(\phi) > 0$, then there exists a Li-Yorke pair for this system.*

Proof. The fact that $h(\phi) > 0$ implies, using Proposition 2.1, that there exists a ϕ -ergodic measure μ , such that

$$h_\mu(\phi) > 0.$$

Let $Z = \text{supp}(\mu)$ and $W = \text{supp}(\lambda_\mu)$. For each non-empty open set U of W in its relative topology, let

$$W_U = \limsup_{n \rightarrow \infty} \phi^{-n}(U).$$

The ergodicity of λ_μ implies that $\lambda_\mu(W_U) = 1$. Therefore, if β is an enumerable basis for the topology of W , then $\lambda_\mu(W_\beta) = 1$ for

$$W_\beta = \bigcap_{U \in \beta} W_U.$$

Together with Lemma 2.8, this implies that

$$\lambda_\mu(W_\beta \setminus \Delta) = 1.$$

In particular, for every point $(a, b) \in W_\beta$, with $a \neq b$, there is a sequence n_k such that $(T^{n_k}a, T^{n_k}b) \rightarrow (a, b)$. And also, by Lemma 2.9, if we pick $c \in Z$, then $(c, c) \in W$. And therefore, there is a sub-sequence m_k such that $(T^{m_k}a, T^{m_k}b) \rightarrow (c, c)$. That is, any such pair (a, b) is a Li-Yorke pair. \square

The following lemma is a simple topological fact.

Lemma 2.11. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a Lie group G . Then,*

$$\phi \text{ is proper} \Leftrightarrow \ker(\phi) \text{ is compact.}$$

Proof. Being ϕ a proper mapping, $\ker(\phi)$ is evidently compact, since it is the inverse image of $\{1_G\}$. On the other hand, a surjective endomorphism is always an open mapping. If $\ker(\phi)$ is compact, then ϕ is a continuous surjection with compact fibers. It is a known fact that any continuous open surjection with compact fibers is a proper mapping. \square

We end this preliminary section with the following characterization of the recurrent set of a linear isomorphism of a finite dimensional vector space proved in Proposition 4.2 in [10].

Proposition 2.12. *Let $T : V \rightarrow V$ be a linear isomorphism of a finite dimensional vector space and consider its multiplicative Jordan decomposition $T = T_E T_H T_U$. The recurrent set of T is given by*

$$\mathcal{R}(T) = \text{fix}(T_H) \cap \text{fix}(T_U).$$

3 The Abelian Case

In this section, we determine the entropy of a surjective endomorphism ϕ of connected abelian Lie group G . In this case, the exponential map is a surjective group homomorphism with discrete kernel. Therefore, G can be identified with $T(G) \times \mathbb{R}^q$, where $T(G)$ is isomorphic to the p -dimensional torus $\mathbb{R}^p/\mathbb{Z}^p$ and the group operation is addition. The endomorphisms of G can be identified with linear maps of the form

$$\phi = \begin{pmatrix} T & * \\ 0 & S \end{pmatrix},$$

where $T : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a linear map that leaves \mathbb{Z}^p invariant, and $S : \mathbb{R}^q \rightarrow \mathbb{R}^q$ is a linear isomorphism. Notice that for $(x, y) \in T(G) \times \mathbb{R}^q$, the action of ϕ have the form

$$\phi^n(x, y) = (*, S^n y).$$

In particular, if (x, y) is a recurrent point, then $y \in \mathcal{R}(S)$ and thus

$$\text{cl}(\mathcal{R}(\phi)) \subset T(G) \times \text{cl}(\mathcal{R}(S)).$$

But since S and S_E coincide in $\text{cl}(\mathcal{R}(S))$, we have that ϕ and

$$\begin{pmatrix} T & * \\ 0 & S_E \end{pmatrix}$$

coincide in $T(G) \times \text{cl}(\mathcal{R}(S))$. And since S_E has only eigenvalues with modulus 1, Bowen's formula and the variational principle gives that

$$h(\phi) \leq h_d(\phi) = h_d(\phi|_{T(G)}) = h(\phi|_{T(G)}) \leq h(\phi),$$

where d is an invariant distance. That is, the topological entropy of ϕ is just the topological entropy of ϕ restricted to its toral component.

4 The Nilpotent Case

4.1 Properness

We start proving that every surjective endomorphism ϕ of a connected nilpotent Lie group G is a proper map. Since the ϕ' is a surjective linear endomorphism of the Lie algebra, it is a linear isomorphism and thus a

proper map. If the G is simply-connected nilpotent, we have ϕ is conjugated to its differential at the identity ϕ' and the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi'} & \mathfrak{g} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\phi} & G \end{array}$$

Thus ϕ is an automorphism and a proper map. We can reduce the general connected nilpotent case to the simply-connected one. But first we need the following lemmas.

Proposition 4.1. *Let G be a nilpotent Lie group and $T(G)$ its toral component. Then $G/T(G)$ is simply connected.*

Proof. Let $\pi : \tilde{G} \rightarrow G$ be the universal covering of G . We have that G is isomorphic to $\tilde{G}/\ker(\pi)$. Besides this, we have that $Z(\tilde{G})$ is isomorphic to a finite dimensional vector space and that

$$Z(G)_0 = Z(G) = \pi \left(Z(\tilde{G}) \right).$$

Therefore $Z(G)_0$ is isomorphic to $Z(\tilde{G})/\ker(\pi)$ and thus $T(G)$ is isomorphic to $\tilde{T}/\ker(\pi)$, where $\tilde{T} = \pi^{-1}(T(G))$ is isomorphic to a vector subspace. On the other hand, we have that

$$\frac{G}{T(G)} \simeq \frac{\tilde{G}/\ker(\pi)}{\tilde{T}/\ker(\pi)} \simeq \frac{\tilde{G}}{\tilde{T}},$$

which shows that $G/T(G)$ is simply connected, since \tilde{G}/\tilde{T} is homeomorphic to the quotient of a vector space by a vector subspace. \square

Proposition 4.2. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a nilpotent connected Lie group G . Then ϕ is a proper map.*

Proof. By Proposition 4.1, we have that $\tilde{G} = G/T(G)$ is simply connected. Besides this, for any surjective endomorphism $\phi : G \rightarrow G$, there is a surjective endomorphism $\tilde{\phi} : \tilde{G} \rightarrow \tilde{G}$ such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ \pi \downarrow & & \downarrow \pi \\ \tilde{G} & \xrightarrow{\tilde{\phi}} & \tilde{G} \end{array}$$

where π is the canonical projection. In fact, since $\phi(T(G))$ is compact and abelian, it is necessarily contained in $T(G)$. Thus we can define

$$\tilde{\phi}(\pi(x)) = \pi(\phi(x)) = \phi(x)T(G).$$

By the above discussion, since \tilde{G} is simply connected, we have that $\tilde{\phi}$ is an automorphism of \tilde{G} . Now, we claim that $\ker(\phi)$ is a closed subset of the compact set $T(G)$. In fact, we have that

$$\begin{aligned} \phi^{-1}(1_G) &\subset \phi^{-1}(T(G)) \\ &= \phi^{-1}(\pi^{-1}(1_{\tilde{G}})) \\ &= \pi^{-1}(\tilde{\phi}^{-1}(1_{\tilde{G}})) \\ &= \pi^{-1}(1_{\tilde{G}}) \\ &= T(G). \end{aligned}$$

Thus we have that $\ker(\phi)$ is compact and, by Lemma 2.11, we have that ϕ is a proper map. \square

4.2 Topological Entropy

In this subsection, we show that, as in the abelian case, the entropy of a surjective endomorphism of a nilpotent Lie group G coincides with the entropy of its restriction to the toral component of G .

Theorem 4.3. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a connected nilpotent Lie group G . Then*

$$h(\phi) = h(\phi|_{T(G)}).$$

Proof. Let $\tilde{G} = G/T(G)$, and denote by $\pi : G \rightarrow G/T(G)$ the canonical projection. By Proposition 4.1, we have that \tilde{G} is a simply-connected nilpotent Lie group and, as in the proof of Proposition 4.2, we can consider the induced endomorphism $\tilde{\phi} : \tilde{G} \rightarrow \tilde{G}$. We have that $\tilde{\phi} : \tilde{G} \rightarrow \tilde{G}$ is conjugated to its differential at the unity $\tilde{\phi}'$ through the exponential map.

On the other hand, we have that $\tilde{\phi}'$ is a linear map and, by Proposition 4.2 in [10], that its recurrent set $\mathcal{R}(\tilde{\phi}')$ is closed. We also know that there is a norm in $\tilde{\mathfrak{g}}$ such that $\tilde{\phi}'|_{\mathcal{R}(\tilde{\phi}')}$ is an isometry. In particular, for any closed ball $B \subset \tilde{\mathfrak{g}}$ centered at 0, $B \cap \mathcal{R}(\tilde{\phi}')$ is compact and $\tilde{\phi}'$ -invariant. From the conjugation given by the exponential map, there is a distance in $\mathcal{R}(\tilde{\phi})$ such that any closed ball $B \subset \mathcal{R}(\tilde{\phi})$ centered at the unit is compact $\tilde{\phi}$ -invariant.

Let $R = \pi^{-1}(\mathcal{R}(\tilde{\phi}))$. Then, since R is closed, it follows that $\text{cl}(\mathcal{R}(\phi)) \subset R$. For any $\varepsilon > 0$, there exists an admissible covering $\mathcal{A} = \{A_0, \dots, A_k\}$ of R , such that

$$h(\phi) - \varepsilon = h(\phi|_R) - \varepsilon \leq h(\phi|_R | \mathcal{A}).$$

This admissible cover can be chosen in a way that A_0 has compact complement, and A_1, \dots, A_k have compact closure, since, in a locally compact space, any admissible covering can be refined in this way. Let then $B \subset \mathcal{R}(\tilde{\phi})$ be a compact $\tilde{\phi}$ -invariant ball such that A_1, \dots, A_k all fall in B . Denoting $K = \pi^{-1}(B)$, it follows that K is compact (since π is proper), and $N_{R \setminus K}(\mathcal{A}^n) = 1$ (for K is ϕ -invariant and $K \subset A_0$). So,

$$\begin{aligned} h(\phi|_R | \mathcal{A}) &\leq h(\phi|_R | \mathcal{A} \vee \{K, K^c\}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log (N_K(\mathcal{A}^n) + N_{R \setminus K}(\mathcal{A}^n)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log (N_K(\mathcal{A}^n) + 1) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log N_K(\mathcal{A}^n) \\ &= h(\phi|_K | \mathcal{A} \cap K). \end{aligned}$$

Therefore,

$$h(\phi) - \varepsilon \leq h(\phi|_R | \mathcal{A}) \leq h(\phi|_K | \mathcal{A} \cap K) \leq h(\phi|_K).$$

On the other hand, applying Proposition 2.5 for the compact $T(G)$ -principal bundle $\pi|_K : K \rightarrow B$, since $\tilde{\phi}$ is an isometry when restricted to B , we conclude that

$$h(\phi|_K) = h(\tilde{\phi}|_B) + h(\phi|_{T(G)}) = h(\phi|_{T(G)}).$$

This way,

$$h(\phi) - \varepsilon \leq h(\phi|_{T(G)}) \leq h(\phi).$$

And since ε was arbitrary, it follows that

$$h(\phi) = h(\phi|_{T(G)}).$$

□

5 The Semi-simple Case

5.1 Properness

We start proving that every surjective endomorphism ϕ of a connected semi-simple Lie group G is a proper map. In fact, we show that, in this case, every surjective endomorphism is an automorphism.

Proposition 5.1. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a semi-simple connected Lie group G . Then there is a $k \in \mathbb{N}$ such that $\phi^k = C_g$ for some $g \in G$, where C_g is the conjugation by g . In particular, ϕ is an automorphism.*

Proof. Notice that

$$\phi^k(\exp X) = \exp [(\phi')^k X] .$$

But since \mathfrak{g} is semi-simple, we know that there is a $k \in \mathbb{N}$ such that $(\phi')^k$ is an internal endomorphism of \mathfrak{g} (see Theorem 5.4, page 423 of [7]). That is, there exists $g \in G$ such that $(\phi')^k = \text{Ad}(g)$ and hence

$$\phi^k(\exp X) = \exp((\phi')^k X) = \exp(\text{Ad}(g)X) = C_g(\exp X).$$

Since G is generated by elements of the form $\exp X$, it follows that $\phi^k = C_g$.

We have that ϕ is an automorphism, since C_g is an automorphism. \square

5.2 Topological Entropy

In this subsection, we use the previous result in order to show that surjective endomorphisms of connected semi-simple Lie groups have zero entropy. Proposition 5.1, shows that some iteration of a surjective endomorphism ϕ of a semi-simple Lie group G is in fact a conjugation by some element of G . Thus we first consider the dynamics of conjugations. For some $g \in G$, we denote by G_g the centralizer of g in G , which is the set of fixed points of the conjugation C_g .

Lemma 5.2. *Let G be a connected linear semi-simple Lie group, and $C_g : G \rightarrow G$ the conjugation by $g \in G$. Then,*

$$\mathcal{R}(C_g) = G_h \cap G_u,$$

where $g = ehu$ is the multiplicative Jordan decomposition of g . In particular, C_g restricted to its recurrent set is an isometry for some distance.

Proof. Notice that $C_g = \text{Ad}(g)|_G$, where Ad is the adjoint representation of $\text{Gl}(d)$, with $G \leq \text{Gl}(d)$. Also, Lemma 3.6 in [11] shows that

$$\text{Ad}(g) = \text{Ad}(e) \text{Ad}(h) \text{Ad}(u)$$

is the Jordan decomposition for $\text{Ad}(g)$.

Since the recurrent set behaves well with respect to restrictions, we know from Proposition 2.12 that

$$\begin{aligned}
\mathcal{R}(C_g) &= \mathcal{R}(\text{Ad}(g)) \cap G \\
&= \text{fix}(\text{Ad}(h)) \cap \text{fix}(\text{Ad}(u)) \cap G \\
&= \text{fix}(C_h) \cap \text{fix}(C_u) \\
&= G_h \cap G_u.
\end{aligned}$$

The last claim follows, since C_g coincides in $\mathcal{R}(C_g)$ with the elliptic linear isomorphism $\text{Ad}(e)$. \square

Theorem 5.3. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a connected semi-simple Lie group G . Then*

$$h(\phi) = h(\phi|_{T(G)}) = 0.$$

Proof. Since $h(\phi^k) = kh(\phi)$, we have that $h(\phi) = 0$ if and only if $h(\phi^k) = 0$. Since G is connected and semi-simple, there is $k > 0$ such that $\phi^k = C_g$, for some $g \in G$. Therefore, it is enough to prove that $h(C_g) = 0$.

From proposition 2.10, we know that, if $h(C_g) > 0$, there exists a Li-Yorke pair for C_g , that is, two distinct elements $a, b \in G$, such that $(C_g^{m_k}(a), C_g^{m_k}(b)) \rightarrow (a, b)$ and $(C_g^{m_k}(a), C_g^{m_k}(b)) \rightarrow (c, c)$, for some $c \in G$. Consider $C_{\text{Ad}(g)}$, and notice that $\text{Ad} \circ C_g = C_{\text{Ad}(g)} \circ \text{Ad}$. Now, since $a, b \in \mathcal{R}(C_g)$, we also have that $\text{Ad}(a), \text{Ad}(b) \in \mathcal{R}(C_{\text{Ad}(g)})$. But $C_{\text{Ad}(g)}|_{\mathcal{R}(C_{\text{Ad}(g)})} = C_{\text{Ad}(e)}|_{\mathcal{R}(C_{\text{Ad}(g)})}$ is an isometry for some distance in $\text{Ad}(G)$. This way, the fact that $(C_{\text{Ad}(g)}^{m_k}(\text{Ad}(a)), C_{\text{Ad}(g)}^{m_k}(\text{Ad}(b)))$ converges to $(\text{Ad}(c), \text{Ad}(c))$ implies that $\text{Ad}(a) = \text{Ad}(b)$.

So, we know that $a = wu$ and $b = wv$ for some $w \in G$ and $u, v \in Z(G)$. We also have that

$$\begin{aligned}
C_g^{m_k}(w)u &= C_g^{m_k}(a) \rightarrow c \\
C_g^{m_k}(w)v &= C_g^{m_k}(b) \rightarrow c.
\end{aligned}$$

But this means that $u = v$. And then $a = b$, contradicting the fact that they are a Li-Yorke pair. \square

6 The Reductive Case

6.1 Properness

Let G be a connected reductive Lie group. It will be useful to consider the surjective group homomorphism

$$\begin{aligned} \pi : Z(G)_0 \times G' &\rightarrow G, \\ (z, g) &\mapsto zg \end{aligned}$$

where $Z(G)_0$ is the identity component of its center, and $G' = [G, G]$ is the derived group which is connected and semi-simple. Also, G and $Z(G)_0 \times G'$ have the same Lie algebra $\mathfrak{z} \times \mathfrak{g}'$, where \mathfrak{g} is the Lie algebra of G , \mathfrak{z} is the Lie algebra of $Z(G)_0$ and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the Lie algebra of G' .

Lemma 6.1. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a connected reductive Lie group G . Then, ϕ induces the surjective homomorphism*

$$\tilde{\phi} = \phi|_{Z(G)_0} \times \phi|_{G'}$$

in $Z(G)_0 \times G'$, such that the following diagram commutes

$$\begin{array}{ccc} Z(G)_0 \times G' & \xrightarrow{\tilde{\phi}} & Z(G)_0 \times G' \\ \pi \downarrow & & \downarrow \pi \\ G & \xrightarrow{\phi} & G \end{array}$$

Proof. It is evident that

$$\phi(Z(G)_0) \subset Z(G)_0,$$

since $\phi(Z(G)_0)$ is a connected subgroup of $Z(G)$ containing the identity, and therefore is a subset of $Z(G)_0$. It happens that

$$\phi(G') \subset G'$$

also holds. But this implies that $\phi'(\mathfrak{z}) = \mathfrak{z}$ and $\phi'(\mathfrak{g}') = \mathfrak{g}'$. And because both $Z(G)_0$ and G' are connected, we have that $\phi|_{Z(G)_0} : Z(G)_0 \rightarrow Z(G)_0$ and $\phi|_{G'} : G' \rightarrow G'$ are surjective.

The commutativity of the above diagram is an immediate consequence of the fact that ϕ is an homomorphism. \square

Proposition 6.2. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a connected reductive Lie group G . If π is a proper map, then ϕ is a proper map.*

Proof. First observe that the endomorphism $\tilde{\phi}$ presented in Lemma 6.1 is proper, since it is the product of two proper maps. In fact, we have that $\phi|_{Z(G)_0}$ and $\phi|_{G'}$ are proper endomorphisms, respectively, by Propositions 4.2 and 5.1. Considering the diagram in Lemma 6.1, we have that, if $K \subset G$ is compact, then $\phi^{-1}(K) = \pi \circ \tilde{\phi}^{-1} \circ \pi^{-1}(K)$ is also compact, since $\tilde{\phi}$ and π are proper maps. \square

6.2 Topological Entropy

We start computing the topological entropy of the endomorphism $\tilde{\phi}$.

Proposition 6.3. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a reductive connected Lie group G and $\tilde{\phi}$ be the associated endomorphism. Then*

$$h(\tilde{\phi}) = h(\phi|_{T(G)}).$$

Proof. Using Proposition 2.2, we have that

$$h(\tilde{\phi}) = h(\phi|_{Z(G)_0}) + h(\phi|_{G'}).$$

The result follows, since, from the abelian and semi-simple cases, we know that $h(\phi|_{Z(G)_0}) = h(\phi|_{T(G)})$ and that $h(\phi|_{G'}) = 0$. \square

Corollary 6.4. *For any surjective endomorphism $\phi : G \rightarrow G$ of a simply-connected reductive Lie group G ,*

$$h(\phi) = h(\phi|_{T(G)}).$$

Proof. Since G is a universal covering and since the Lie algebras of G and $Z(G)_0 \times G'$ coincide, the homomorphism π is a conjugation between $\tilde{\phi}$ and ϕ . \square

Now we consider the case where G is not homeomorphic to $Z(G)_0 \times G'$.

Proposition 6.5. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a reductive connected Lie group G . If the projection $\pi : Z(G)_0 \times G' \rightarrow G$ is proper, then*

$$h(\phi) = h(\phi|_{T(G)}).$$

Proof. Consider the endomorphism $\tilde{\phi}$ from Lemma 6.1. Now, use Proposition 2.3 and Proposition 6.3 to conclude that

$$h(\phi) \leq h(\tilde{\phi}) = h(\phi|_{T(G)}) \leq h(\phi).$$

\square

As an immediate consequence, we solve the linear reductive case.

Corollary 6.6. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a connected linear reductive Lie group G . Then,*

$$h(\phi) = h(\phi|_{T(G)}).$$

Proof. Using Proposition 6.5, we just have to show that π is proper. But $\ker(\pi) = \{(x, x^{-1}) \mid x \in Z(G)_0 \cap G'\}$. And, for a linear reductive Lie group G , $Z(G)_0 \cap G'$ is contained in the center of G' . But the center of a linear semi-simple Lie group is always finite. Now, we just use Lemma 2.11. \square

Another immediate consequence is that the formula for the entropy of a endomorphism of a compact group reduces to the formula for the entropy of an endomorphism of a torus.

Theorem 6.7. *Let $\phi : G \rightarrow G$ be a surjective endomorphism of a compact connected Lie group G . Then,*

$$h(\phi) = h(\phi|_{T(G)}).$$

Proof. Since every compact Lie group is a reductive Lie group (see Proposition 6.6, page 132 of [7]), we just have to show that π is proper. But $Z(G)_0$ and G' are compact subgroups of the compact group G (see Theorem 6.9, page 133 of [7]). Then $Z(G)_0 \times G'$ is compact and π is proper. \square

7 Remarks on the General Case

In this section, we present a conjecture about the entropy of surjective proper endomorphisms ϕ of a connected Lie group G . Based on the previous particular cases, one may conjecture that the entropy ϕ coincides with the entropy of its restriction to the toral component of G . But, if we could prove Proposition 2.5 for locally compact principal bundles, we would first conclude that

$$h(\phi) = h(\phi|_{R_0}),$$

where $R = R(G)_0$ is identity component of the solvable radical $R(G)$ of G . In fact, considering the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ \pi \downarrow & & \downarrow \pi \\ G/R & \xrightarrow{\tilde{\phi}} & G/R \end{array},$$

which is well defined, since $\phi(R) = R$, we would have that

$$h(\phi) = h(\phi|_R) + h(\widetilde{\phi}).$$

Thus we get the above first formula, since G/R is a connected semi-simple Lie group. Now, considering R' , the derived subgroup of R , and the diagram

$$\begin{array}{ccc} R & \xrightarrow{\phi|_R} & R \\ \pi \downarrow & & \downarrow \pi \\ R/R' & \xrightarrow{\widetilde{\phi|_R}} & R/R' \end{array},$$

which is well defined, since $\phi(R') = R'$, we would have that

$$h(\phi|_R) = h(\phi|_{R'}) + h(\widetilde{\phi|_R}).$$

Since R' is connected nilpotent and R/R' is connected abelian, putting all together, we would have that

$$h(\phi) = h(\phi|_{T(R)}) + h(\widetilde{\phi|_R}|_{T(R/R')}).$$

We would conclude that the formula of the topological entropy of a surjective proper endomorphism of a connected Lie group would reduce to the formula of the topological entropy of a surjective endomorphism of a torus.

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